I. INTRODUCTION

The theoretical investigation of hyperbranched polymers \(^1\)-\(^{12}\) is enjoying growing attention. Up to now, the analytic work focused on regular hyperbranched polymers such as dendritic structures (DS) \(^2\), \(^6\), \(^{13}\)-\(^{18}\) and Vicsek fractals (VF) \(^7\)-\(^{11}\), \(^{19}\)-\(^{24}\). While DS encounter a packing problem with increasing generation (which, in fact, limits their growth), VF are considerably more sparse and hence can, in principle, grow indefinitely. \(^10\) Furthermore, in the standard extension of the Rouse model \(^1\), \(^{23}\), \(^{26}\) in the so-called generalized Gaussian scheme (GGS), \(^1\) several dynamical properties of VF, such as the mechanical relaxation moduli, scale, \(^7\)-\(^{11}\) a finding distinct from the behavior of DS.

Now, the GGS depicts fully flexible macromolecules and is thus not particularly realistic. As shown by us in previous work, \(^7\) in models for treelike polymers, TP (both the DS and the VF belong to this class), one can introduce analytically restrictions on the orientations of the bonds and hence mimic semiflexibility. For semiflexible treelike polymers (STP), this procedure is a natural extension of previously advanced methods for treating semiflexible chains.\(^{28}\), \(^{29}\)

In this paper we focus on the determination and the solution of the equations of motion (EM) of semiflexible Vicsek fractals (SVF). Because of the angular restrictions, the SVF-problem is much more complex than the VF-problem. Namely, the full spectrum of eigenvalues of the latter can be determined (for arbitrary functionalities and generations) to very high accuracy through an iterative scheme.\(^7\)-\(^{11}\), \(^{19}\)-\(^{22}\) To SVF this scheme does not apply; we can, nonetheless, obtain the full SVF-spectrum by using a judicious method: It consists in extending the procedure developed in treating semiflexible dendritic structures (SDS), the basic idea being the iterative construction of a complete set of eigenvectors, \(^{30}\) from which all the eigenvalues and their degeneracies can be determined. Both for SDS and for SVF the procedure is rendered possible due to the high topological symmetry of these regular hyperbranched polymers.

The paper is structured as follows: In Sec. II we recall the topological properties of DS and of VF. Section III presents the extension of the GGS-model to the STP-framework, by which the semiflexibility is taken into account. In Sec. IV we present the different elements entering the EM for SVF. This allows us to construct in Sec. V complete sets of eigenmodes for SVF of arbitrary functionality and generation; these reduce considerably the EM of SVF and allow to analytically determine the degeneracies of the corresponding eigenvalues. Section VI shows several examples, while the general scheme is deferred to the appendices. In Sec. VII we discuss the mechanical loss moduli \(G''(\omega)\) for SVF of functionality \(f = 4\) and generation \(g = 8\) and compare these results to those found for semiflexible DS. Finally, Sec. VIII ends with our conclusions.

II. REGULAR HYPERBRANCHED MACROMOLECULES

Hyperbranched polymers are of much interest in modern macromolecular science.\(^{31}\) Their characteristic feature is that many of their junctions are branched. While batch reactions often produce random structures, for theoretical purposes one focuses on regular (i.e., deterministically built) hyperbranched polymers.

Typical for this class of regular hyperbranched macromolecules are the DS and the VF. In particular, dendrimers, a special case of DS, have attracted much attention. In Fig. 1 we recall their structure by using exemplarily a fourfold functionality, \(f = 4\). The fundamental feature here is that all the beads, except for the peripheral ones, are directly connected to four nearest neighbors (NN). Other DS are the so-called “wedges”, as discussed in Refs. 5 and 6. In the case of wedges the central monomer, the “core,” has a different functionality than the other internal beads.
FIG. 1. Schematic drawing of a dendrimer of functionality $f = 4$ and generation $g = 3$. The dashed circles depict the three different generations, see text for details. Note that the picture is only meant to exemplify the connections between the beads. A particular snapshot of the dendrimer may show a vastly different geometric form.

Another class of deterministically built hyperbranched structures is represented by the VF. The construction of a VF proceeds iteratively, see Fig. 2: A VF of generation $g = 1$ consists of a central bead (the core) to which $f$ other beads are attached. Then, a VF of $g = 2$ is built from $(f + 1)$ VF of generation $g = 1$: A central VF acts as a seed and the other $f$ VF are attached to it as depicted in Fig. 2. The procedure is then iterated; thus Fig. 2 extends up to $g = 3$. It follows that a VF of functionality $f$ and generation $g$ consists of

$$N = (f + 1)^g$$

beads. In Table I we present exemplary some $N$-values for different $f$ and $g$.

It is worthwhile to recall that the VF are self-similar, with fractal dimension $d_r = \ln (f + 1)/\ln 3$. Hence, for all chemically reasonable values of $f$ (for these $d_r < 3$) and for all $g$, the VF can be readily embedded in the three-dimensional space. This is distinct from the DS, whose maximally achievable $g$-values are limited because of overcrowding. Moreover, changing $f$ one can study VF of different fractal dimensions.

Given their properties, VF have been thoroughly studied in several fields. An important feature of the VF is their dynamic scaling behavior. Scaling is fundamental in polymer physics and is also displayed by other hierarchically built structures, but not by DS. We note that in order to study scaling in our framework, we have to determine the full eigenvalue spectrum of very large structures. Calculating this spectrum based on the EM of SVF is one of the aims of the present work. As we proceed to show, the symmetry of the VF is of considerable practical help in this endeavor.

III. MODELLING SEMIFLEXIBILITY

In this section we recall the basic approach for describing semiflexible behavior. First, the structure is represented through beads, whose positions are given by the set $\{r_i\}$. Then one introduces the bonds, so that, say, sites $r_i$ and $r_j$ are directly connected by the bond $d_{ij}$, hence $d_{ij} = r_i - r_j$. When the bonds are harmonic and in the absence of additional restrictions the system is fully flexible; it represents a GGS.

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TABLE I. Number of beads $N$ in a VF of functionality $f$ and generation $g$. 

FIG. 2. Schematic drawing of Vicsek fractals of functionality $f = 4$ and generations $g = 1, 2, \text{and 3}$. The dashed drop-like substructures are leaves of generations 1, 2, and 3 connected by a single bond to the VF, see text for details. We also indicate by a solid line a substructure connected by two bonds to the VF. Note that the picture is only meant to exemplify the connections between the beads. Particular snapshots of the Vicsek fractals may come up in vastly different geometric forms.
whose potential energy is
\[ V_{\text{GGS}} ([\mathbf{d}_a]) = \frac{K}{2} \sum_a \mathbf{d}_a^2. \] (2)

In Eq. (2) the sum runs over all the bonds \([\mathbf{d}_a]\), the spring constant \(K\) equals \(3k_BTl^2\), where \(k_B\) denotes the Boltzmann constant and \(T\) the temperature, and \(l^2\) is the mean-square bond length, taken to be the same for all bonds.

Restrictions concerning the angles between bonds are then introduced by extending Eq. (2) to
\[ V_{\text{STP}} ([\mathbf{d}_a]) = \frac{K}{2} \sum_{a,b} W_{ab} \mathbf{d}_a \cdot \mathbf{d}_b. \] (3)

A particularly transparent way to do so holds for treelike structures, i.e., for polymers devoid of loops. In this case one can independently specify angular restrictions at every junction, say for two adjacent bonds, \(a\) and \(b\), having the bead \(i\) in common, by setting \(\mathbf{d}_a \cdot \mathbf{d}_b = \pm l^2 q_i\). Here \(q_i\) denotes the degree of stiffness, and the plus sign holds for a head to tail orientation of \(\mathbf{d}_a\) and \(\mathbf{d}_b\) and the minus sign otherwise, see Refs. 18 and 27 for details. Interestingly, both the freely rotating segment condition (as suggested by Bixon and Zwanzig for semiflexible chains)\(^{28}\) and the maximum entropy principle\(^{29}\) lead to the same relations for the \([\mathbf{d}_a \cdot \mathbf{d}_b]\)-values for all non-adjacent pairs of bonds \((\mathbf{d}_a, \mathbf{d}_b)\).\(^{18,27}\) All these expressions determine then the elements of the matrix \(W = (W_{ab})\), see Refs. 18 and 27.

The dynamics of the STP follows from Eq. (3) by using the Langevin equation framework,\(^{1,26}\) thus the equation of motion for the \(x\)-component of the position vector \(\mathbf{r}_i\) reads
\[ \zeta \frac{\partial}{\partial t} x_i(t) + \frac{\partial}{\partial x_i} V_{\text{STP}} ([\mathbf{r}_i]) = \tilde{f}_i(t), \quad \text{for all } i. \] (4)

In Eq. (4) \(\tilde{f}_i\) is the \(x\)-component of the Gaussian stochastic force \(\tilde{f}\) acting on the \(i\)th bead, for which \(\langle \tilde{f}_i(t) \rangle = 0\) and \(\langle \tilde{f}_i(t) \tilde{f}_j(t') \rangle = 2k_BT \zeta \delta_{ij} \delta(t-t')\) hold.

Note that through Eq. (3) \(V_{\text{STP}}\) depends on the bond variables \([\mathbf{d}_a]\). Using relations such as \(\mathbf{d}_a = \mathbf{r}_i - \mathbf{r}_j\), i.e., in general
\[ \mathbf{d}_a = \sum_i (\mathbf{G}^T)_{ai} \mathbf{r}_i, \] (5)
where \(\mathbf{G}\) is the so-called incidence matrix\(^{30}\) and \(T\) denotes the transposed, one obtains as a function of the bead variables
\[ V_{\text{STP}} ([\mathbf{r}_i]) = \frac{K}{2} \sum_{i,j} (\mathbf{A}_{\text{STP}})_{ij} \mathbf{r}_i \cdot \mathbf{r}_j, \] (6)
where \(\mathbf{A}_{\text{STP}} \equiv \mathbf{GWGW}^T\). Based on \(\mathbf{A}_{\text{STP}}\) Eq. (4) reads
\[ \tau_0 \frac{\partial}{\partial t} x_i(t) + \sum_{j=1}^N A_{\text{STP}}^{ij} x_j(t) = \tilde{f}_i(t)/K, \quad \text{for all } i, \] \[ \text{with } \tau_0 = \zeta/K. \] (7)

Now, for STP all the elements of the matrix \(\mathbf{A}_{\text{STP}}\) are known analytically.\(^{18,27}\) In fact, the only non-vanishing entries of \(\mathbf{A}_{\text{STP}}\) are the diagonal elements and the ones involving for each site \(i\) its nearest neighbors (NN), denoted by \(i_k\) (one index), and its next-nearest neighbors (NNN), denoted by \(i_k\) (two indices). The situation is shown in Fig. 3 and the non-vanishing \(A_{ij}^{\text{STP}}\) are\(^{37}\)
\[ A_{ii}^{\text{STP}} = \frac{\phi_i}{1 - (\phi_i - 1)q_i} + \frac{1}{1 - (\phi_i - 2)q_i} \] \[ - \frac{(\phi_i - 1)q_i}{1 - (\phi_i - 1)q_i} \] \[ - \frac{(\phi_i - 1)q_i}{1 - (\phi_i - 1)q_i}, \] \[ A_{kk}^{\text{STP}} = \frac{q_i}{1 - (\phi_i - 1)q_i} \] \[ - \frac{q_i}{1 - (\phi_i - 1)q_i} \] \[ - \frac{q_i}{1 - (\phi_i - 1)q_i}. \] (8)

In Eqs. (8)–(10), \(\phi_i\) and \(\phi_k\) are the functionalities and \(q_i\) and \(q_k\) are the stiffness parameters of the corresponding beads.

In the fully flexible limit \(q_i \to 0\) and \(q_k \to 0\), the matrix \(\mathbf{A}_{\text{STP}}\) reproduces the connectivity (Laplacian) matrix \(\mathbf{A}_{\text{GGS}} = (A^{GGS})_{ij}\) of the GGS-model.\(^{1}\) This matrix is highly sparse, in fact it is given by
\[ A_{ij}^{\text{GGS}} = \phi_i \] \[ \text{and } \] \[ A_{ij}^{\text{GGS}} = \begin{cases} -1, & \text{if } i \text{ and } j \text{ are NN}, \\ 0, & \text{otherwise}. \end{cases} \] (12)

It is worthwhile to recall that the matrices \(\mathbf{A}_{\text{GGS}}\) and \(\mathbf{A}_{\text{STP}}\) do not take excluded volume interactions into account. This fact considerably simplifies the description of the dynamics, but precludes treating effects such as the overcrowding of SD with increasing generation.

Now, solving the set of Langevin equations of Eq. (7) requires the diagonalization of \(\mathbf{A}_{\text{STP}}\). Needed is the complete set of eigenvalues \(\{\lambda_i\}\) of \(\mathbf{A}_{\text{STP}}\). Its knowledge paves the way for the determination of many dynamical characteristics of

FIG. 3. Neighborhood situation of a bead \(i\) in a treelike network. Presented are the nearest and next-nearest neighbors. Here, one of the nearest neighbors of \(i\) is denoted by \(i_k\) and one of the next-nearest neighbors of \(i\) by \(i_k\), see text for details.
polymers. In this paper we focus on mechanical relaxation and hence on the complex shear modulus \( G'(\omega) = G(\omega) + i G''(\omega) \). Denoting the (single) vanishing eigenvalue of \( \mathbf{A}^{\text{STP}} \) by \( \lambda_1, \lambda_1 = 0 \), the dimensionless storage \( G'(\omega) \) and loss \( G''(\omega) \) moduli read

\[
G'(\omega) = \frac{G'(\omega)}{v k_B T} = \frac{1}{N} \sum_{k=2}^{N} \frac{(\omega \tau_0 / 2 \lambda_k)^2}{1 + (\omega \tau_0 / 2 \lambda_k)^2} \quad (13)
\]

and

\[
G''(\omega) = \frac{G''(\omega)}{v k_B T} = \frac{1}{N} \sum_{k=2}^{N} \frac{\omega \tau_0 / 2 \lambda_k}{1 + (\omega \tau_0 / 2 \lambda_k)^2} \quad (14)
\]

Equations (13) and (14) show explicitly that for their evaluation the knowledge of the full spectrum, i.e., of all of the \( \lambda_k \), is needed.

**IV. STRUCTURE OF \( \mathbf{A}^{\text{STP}} \) FOR SVF**

Given the results of Sec. III we can now determine the different elements of the matrix \( \mathbf{A}^{\text{STP}} \) for SVF, matrix which we denote by \( \mathbf{A}^{\text{SVF}} \). We remark that \( \mathbf{A}^{\text{SVF}} \) has more non-vanishing entries than \( \mathbf{A}^{\text{STP}} \), the \( \mathbf{A}^{\text{GGG}} \)-matrix of the fully flexible VF. Namely, distinct from \( \mathbf{A}^{\text{STP}} \), the matrix \( \mathbf{A}^{\text{SVF}} \) contains also elements corresponding to NNN-sites.

For simplicity we assume that all VF junctions of the same functionality \( \phi \) have the same stiffness degree. From Fig. 2 it is evident that in a VF the beads have as \( \phi \)-values 1, 2 or 3. Given that beads of functionality \( \phi = 1 \) do not correspond to junctions, we have to consider only two distinct stiffness degrees, namely \( q_2 \) and \( q_6 \). Moreover, we will call the beads with \( \phi = 1 \) dangling beads. It is worthwhile to note that in dendrimers only beads with \( \phi = 1 \) and \( \phi = f \) appear, see Fig. 1. This renders the procedure for SVF more demanding than for SDS.

From Eqs. (8) to (10) we now determine the different elements of \( \mathbf{A}^{\text{SVF}} \); depending on the particular topological neighborhoods of each particular site, the distinct situations are presented in Fig. 4.

First, there are exactly six distinct possibilities for the diagonal elements \( \mathbf{A}^{\text{SVF}}_{ii} \), for which all the NN of each particular site and their connections are displayed:

1. If \( i \) is a dangling bead, \( \phi = 1 \), it is connected via a single neighbor of functionality \( \phi = f \) to the rest of the SVF. Thus, from Eq. (8) the corresponding value of \( A^{\text{SVF}}_{ii} \) follows (we denote it by \( \mu_1 \))

\[
\mu_1 = 1 + \frac{(f - 1) q_f}{1 - (f - 2) q_f - (f - 1) q_f} \quad (15)
\]

2. A VF of generation \( g \leq 1 \) is a star in which the central bead \( c \) has as neighbors only \( f \) dangling beads. From Eq. (8) \( A^{\text{SVF}}_{cc} \) equals then

\[
\mu_2 = \frac{f}{1 - (f - 1) q_f} \quad (16)
\]

The following four situations occur only for VF of generation \( g > 1 \).

3. If a bead, say \( j \), has \( (f - 1) \) dangling beads as NN, then it has one bead of functionality \( \phi = 2 \) as NN too. Hence, based on Eq. (8), \( A^{\text{SVF}}_{jj} \) equals

\[
\mu_3 = \frac{f}{1 - (f - 1) q_f} + \frac{q_2^2}{1 - q_2^2} \quad (17)
\]

4. A bead of functionality \( \phi = 2 \), say \( k \), has always one NN for which \( \phi = f \) and one NN for which \( \phi = 2 \). Thus from Eq. (8) \( A^{\text{SVF}}_{kk} \) is

\[
\mu_4 = \frac{2}{1 - q_2} + \frac{(f - 1) q_f^2}{1 - (f - 2) q_f - (f - 1) q_f} + \frac{q_2^2}{1 - q_2^2} \quad (18)
\]

5. Now, if a bead with \( \phi = f \), say \( m \), has only \( (f - 2) \) dangling NN, then another two of its NN beads are of functionality \( \phi = 2 \). In this case Eq. (8) gives for \( A^{\text{SVF}}_{mm} \):

\[
\mu_5 = \frac{f}{1 - (f - 1) q_f} + \frac{2 q_2^2}{1 - q_2^2} \quad (19)
\]

**FIG. 4.** Schematic drawing of the non-vanishing matrix elements of \( \mathbf{A}^{\text{SVF}} \). Here beads of functionality \( \phi = 1 \) are indicated by full black circles, beads of functionality \( \phi = 2 \) by full green circles, while beads of functionality \( \phi = f \) (here \( f = 4 \)) are illustrated by full red circles. The open circles denote the beads for which the diagonal elements take particular \( \mu \)-values; the wavy lines indicate the non-diagonal terms as interactions: \( v_1, v_2, \) and \( v_3 \) for the NN, \( \rho_1 \) and \( \rho_2 \) for the NNN beads.
6. Finally, bead \( n \) may be well inside the VF, in which case all its first NN have the functionality \( \phi = 2 \). Then from Eq. (8) \( A_{ii}^{\text{SVF}} \) equals
\[
\mu_6 = \frac{f}{1 - (f - 1)q_f} + f \frac{q_2}{1 - q_2^2}. \tag{20}
\]

As a second step we consider the non-diagonal NN elements of \( A^{\text{SVF}} \), namely the \( A_{ij}^{\text{SVF}} \). There are three distinct possibilities, as depicted in Fig. 4; we note that beads of functionality \( \phi = f \) can have as neighbors only beads of functionality \( \phi = 1 \) and \( \phi = 2 \).

1. A dangling bead, say \( i \), can have as neighbor only a bead for which \( \phi = f \). From Eq. (9) this implies for \( A_{ii}^{\text{SVF}} \):
\[
\nu_1 = \frac{1}{1 - (f - 1)q_f}. \tag{21}
\]

2. For a bead \( i \) with \( \phi = 2 \) we have to distinguish between its two neighbors. If \( i_k \) has \( \phi = f \), then using Eq. (9) \( A_{ii}^{\text{SVF}} \) equals
\[
\nu_2 = \frac{1 - (f - 1)q_2q_f}{(1 - (f - 1)q_f)(1 - q_2)}. \tag{22}
\]

3. Otherwise, \( i_k \) is also of functionality \( \phi = 2 \), so that \( A_{ii}^{\text{SVF}} \) turns out to be
\[
\nu_3 = \frac{1 + q_2}{1 - q_2}. \tag{23}
\]

Finally, we have to consider the NNN elements \( A_{ij}^{\text{SVF}} \). These elements, as it follows from Eq. (10), depend only on the parameters of the mutual NN \( i_k \) of \( i \) and \( i_k \). Hence \( A_{ij}^{\text{SVF}} \) depends on the functionality \( \phi \) of \( i_k \):

1. For \( \phi = f \), \( A_{ij}^{\text{SVF}} \) equals
\[
\rho_1 = \frac{q_f}{1 - (f - 2)q_f - (f - 1)q_f^2}. \tag{24}
\]

2. For \( \phi = 2 \), \( A_{ij}^{\text{SVF}} \) is
\[
\rho_2 = \frac{q_2}{1 - q_2^2}. \tag{25}
\]

All other non-diagonal elements of the matrix \( A^{\text{SVF}} \) are zero.

V. CHOICE OF AN ADEQUATE SET OF EIGENVECTORS FOR \( A^{\text{SVF}} \)

After having discussed the structure of the matrix \( A^{\text{SVF}} \), we now turn to the determination of a complete set of eigenvalues for it, which allows a semi-analytical computation of its eigenvalues and of their degeneracies. Here we make a judicious use of the symmetry of the underlying VF, by extending the ideas developed for dendrimers in Refs. 2 and 30.

We start by introducing subsets of the VF, which we call “leaves” and denote them by \( \mathcal{L} \). In Fig. 2, which shows a VF with \( f = 4 \), we indicate through dashed lines in the shape of a drop three leaves of generation 1, 2, and 3. Leaf \( \mathcal{L}^{(1)} \) of generation 1 consists of a single bead. Leaves of higher generation can be built iteratively following the procedure indicated in Fig. 5: Leaf \( \mathcal{L}^{(g)} \), of generation \( g \), is formed from \( (f + 1) \) leaves \( \mathcal{L}^{(g-1)} \), of which \( f \) are connected to a central, “glue” monomer and the remaining \( \mathcal{L}^{(g-1)} \) leaf is connected back-to-back to this object, see Fig. 5. Each leaf has a bead through which it is connected to the rest of the VF; it is located at the top of the drop and we indicated it in red in Fig. 5. In the further text we call this bead “root bead” of the leaf.

The matrix \( A^{\text{SVF}} \) leads, based on Eq. (7), to the following general set of homogeneous EM:
\[
\tau_0 \dot{x}_i + \sum_{j=1}^{N} A_{ij}^{\text{SVF}} x_j = 0 \quad \text{for } 1 \leq i \leq N. \tag{26}
\]

Focusing on leaves helps now to limit the number of non-vanishing \( x_j \) in Eq. (26). For this we recall that the \( A_{ij}^{\text{SVF}} \) vanish for all pairs \((i, j)\) that are further apart than in NNN position.

As an example, we start from two red (dangling) beads, \( i \) and \( j \), in NNN position in a SVF of generation \( g \geq 2 \), see Fig. 6(a). It is a simple matter to verify that one eigenmode of \( A^{\text{SVF}} \) is given by
\[
x_i = -x_j \quad \text{and} \quad x_k = 0 \quad \text{for all } k \text{ such that } i \neq k \neq j. \tag{27}
\]

It suffices namely, due to the above-mentioned property of \( A^{\text{SVF}} \), to consider, apart from the equations of motion for \( x_i \) and \( x_j \), only those for the beads which are NN or NNN to \( i \) or \( j \). Then the system of Eqs. (26) reduces, using Eqs. (15) and (24), to a single non-trivial equation:
\[
-\tau_0 \dot{x}_i = \mu_1 x_i - \rho_1 x_i = (\mu_1 - \rho_1) x_i, \tag{28}
\]
from which the eigenvalue
\[
\lambda = \mu_1 - \rho_1 \tag{29}
\]
follows, when the time is measured in units of \( \tau_0 \). The motion described by Eq. (27) is that of two leaves (here \( \mathcal{L}^{(1)} \), i.e., very simple ones) that move against each other.

The next example concerns two \( \mathcal{L}^{(2)} \) leaves in NNN position, as depicted in Fig. 6(b). Here the motion is such that in each leaf beads that are equivalent by symmetry (denoted by blue, black, green, and red) have the same amplitude, which
is exactly opposite to that of the corresponding beads of the other leaf. Again it is a simple matter to show that such a mode allows the other beads of the VF to stay at rest. The detailed analysis of the motion of Fig. 6(b) and of larger $L^{(g)}$ leaves will be discussed in Sec. VI and in the appendices. We remark that the mode discussed here gives rise to four new eigenvalues (corresponding to the four colors).

The procedure outlined can be iterated to leaves of higher order. It is important to note that the lower order leaves lead to the most degenerate eigenvalues, which hence carry most of the weight in determining the relaxation modulus, Eq. (14).

Now, we proceed by showing that through this procedure we obtain for each eigenvalue its exact degeneracy and that, taking these degeneracies into account, we indeed find all the eigenvalues of the SVF-spectrum.

Grouping the eigenmodes by the generation of the leaves $L$ involved, we observe that a VF of generation $g$ displays $(g + 1)$ different groups: In the first group two $L^{(1)}$ leaves in NNN position move against each other, in the second group two NNN $L^{(2)}$ leaves do so. In general, in group $n, n \leq g$, two NNN $L^{(n)}$ leaves move against each other. Finally, in group $(g + 1)$, all beads of the VF, including the core of the VF, move.

Let us focus now on the $n$th group, in which two NNN leaves $L^{(n)}$ move against each other, while in each leaf beads which are equivalent by symmetry have the same amplitude. Each leaf $L^{(n)}$ consists of $(f + 1)$ connected $L^{(n-1)}$ leaves, as illustrated in Fig. 7(a) for $f = 4$. Moreover, to render clear which beads are related by symmetry, we color the leaves following the pattern of Fig. 6(b): The $(f - 1)$ external leaves are blue and we denote them by $L^{(n-1)}_1$, the other two are green and red and we denote them by $L^{(n-1)}_2$ and by $L^{(n-1)}_3$, respectively, the glue bead being again black. The index, 1 or 2, counts the number of bonds that connect the given $L^{(n-1)}$ to the VF.

For general $n$ we can now readily determine, based on the leaves $L^{(n)}$, the number of independent non-vanishing variables $x_i$ that enter the corresponding EM. Clearly, there are two types of leaves, namely those connected through one and those connected through two bonds to the rest of the VF, as depicted in Figs. 7(a) and 7(b), respectively. We denote by $F_1(n)$ and by $F_2(n)$ their corresponding number of independent variables. From Fig. 7(a) one reads simply off that for a $L^{(n)}_1$

$$F_1(n) = F_1(n - 1) + 2F_2(n - 1) + 1$$

(30)

holds. Furthermore, from Fig. 7(b) one has for a $L^{(n)}_2$

$$F_2(n) = F_1(n - 1) + 3F_2(n - 1) + 1.$$  

(31)

From Eqs. (30) and (31) it follows that

$$F_1(n) = F_2(n) - F_2(n - 1),$$

(32)

whose corollary is

$$F_2(n) = \sum_{i=1}^{n} F_1(i).$$

(33)

Rewriting Eq. (31) for $(n - 1)$ one has

$$F_2(n - 1) = F_1(n - 2) + 3F_2(n - 2) + 1$$

(34)

and subtracting Eq. (34) from Eq. (31) leads to

$$F_1(n) = 4F_1(n - 1) - F_1(n - 2),$$

(35)

where use was made of Eq. (32). Equation (35) allows to determine the $F_1(n)$ iteratively, starting from the values $F_1(1) = 1$ and $F_1(2) = 4$, as discussed above.

Here it helps to recall that the Chebyshev polynomials of the second kind, $U_n(x)$, obey

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

(36)

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, see Eq. (1.6) of Ref. 40. Comparing Eq. (35) to Eq. (36) and the respective initial conditions, allows us to conclude that $F_1(n) = U_{n-1}(2)$. Now, $U_{n-1}(2)$ is known in closed form, see Eq. (1.52) of Ref. 40, so that

$$F_1(n) = \frac{(2 + \sqrt{3})n - (2 - \sqrt{3})n}{2\sqrt{3}}$$

(37)

for $1 \leq n \leq g$.

One may remark that Eq. (37) leads to integer $F_1(n)$ values, despite of the fact that it involves roots of three. Since $(2 + \sqrt{3}) > 1$ whereas $(2 - \sqrt{3}) < 1$, for $n \geq 5$ one can approximate $F_1(n)$ very well by $(2 + \sqrt{3})n/2\sqrt{3}$. 

FIG. 6. Examples of displacements involving leaves whose beads move against each other. Case (a) shows the motion of two $L^{(1)}_1$ leaves and case (b) shows the motion of two $L^{(1)}_2$ leaves.
Returning now to the SVF of generation $g$, we recall that $F_1(n)$ is the number of independent variables in each group $n$, (with $1 \leq n \leq g$), in which two NNN leaves of generation $n$ move against each other.

An additional, independent set of EM arises from the group $(g + 1)$, in which all the beads, including the core, move and beads related by symmetry have identical amplitudes. Taking the core into account, the number of independent variables in this group is $F_1(g) + 1$.

As an example for this motion we consider a SVF of generation $g = 1$, see Fig. 2. This SVF has only two kinds of beads, the peripheral beads and the core. Denoting the amplitude of their motion by $x_1$ and $x_2$, respectively, we have from Eq. (26):

$$
\begin{align*}
-t_0 x_1 &= \mu_1 x_1 + (f - 1) \rho_1 x_1 + v_1 x_2, \\
-t_0 x_2 &= \mu_2 x_2 + f v_1 x_1.
\end{align*}
$$

This system of equations leads to $F_1(1) + 1 = 2$ eigenvalues.

We turn now to the determination of the degeneracies of the eigenvalues, by counting the linearly independent eigenvectors. Denoting by $D(n)$ the degeneracy in the $n$th group we claim that the following holds:

$$
D(n) = \begin{cases} 
1 & \text{for } n = g + 1, \\
\frac{f - 1}{(f + 1)^{g-1-n} [f(f - 2) - 2] + 2} & \text{for } n \leq g - 1.
\end{cases}
$$

In doing so, case $n = g + 1$ corresponds to the unique modes in which the core moves. In case $n = g$ the eigenmodes are such that out of $f$ leaves $L_1^{(g)}$, two move against each other; obviously there are $(f - 1)$ linearly independent choices to pick such $L_1^{(g)}$ pairs. The case $n \leq g - 1$ appears for the first time for $g = 2$. For $g = 2$ the VF consists of $f$ leaves $L_1^{(2)}$, each with $(f - 1)$ dangling beads. The number of linearly independent eigenvectors is then $f[f(f - 1) - 1] = f(f - 2)$, in accordance to Eq. (39).

We now prove Eq. (39) iteratively, for increasing $g$. Each VF of generation $(g + 1)$ is formed by $(f + 1)$ VF of generation $g$, connected to each other by $f$ new bonds. While the whole structure grows thus by a factor of $(f + 1)$, each newly created bond leads for all $n$ to the disappearance of two $L_1^{(n)}$, by which it removes from every set of eigenvectors two of them. The step $g \to g + 1$ changes thus the value of $D(n)$ from $(f + 1)^{g-1-n} [f(f - 2) - 2] + 2$ to $(f + 1)(f + 1)^{g-1-n} [f(f - 2) - 2] + 2(f + 1) - 2f = (f + 1)^{g-n} [f(f - 2) - 2] + 2$, fact which proves our claim.

We now obtain the total number $\mathcal{N}$ of states in the spectrum by combining the contributions from the groups $(1 \leq n \leq g)$, Eqs. (37) and (39), with those from the group $(g + 1)$. Using the expression $\sum_{j=1}^{g} x_j = (x^g - 1)/(x - 1)$ we find after some straightforward algebra that

$$
\mathcal{N} = \sum_{n=1}^{g} D(n)F_1(n) + (F_1(g) + 1) = (f + 1)^{g} = \mathcal{N},
$$

where the last relation follows from Eq. (1). Thus $\mathcal{N}$ equals the total number of beads of the VF, proving that indeed our procedure leads to the full $A_{SVF}$-spectrum.

We conclude by noticing that the complete set of eigenmodes discussed here makes heavy use of the particular symmetry of the SVF and hence cannot be directly applied to other structures. Nevertheless, the ideas put forward here may pave the way for investigating other complex, but still symmetric treelike architectures.
VI. SVF EQUATIONS OF MOTION AND REDUCED MATRICES

After having established the special set of SVF eigenvectors in Sec. V, we use it here to reduce the SVF set of EM given by Eq. (26).

We start with the EM involving two NNN leaves moving against each other. The EM for the group \( n = 1 \) is Eq. (28), as discussed above. Consider now the case \( n = 2 \): Here two \( L_1^{(2)} \) NNN leaves move against each other, as depicted in Fig. 6(b). Using the color code of Fig. 6(b) we call the non-vanishing amplitudes \( x_1 \) (blue), \( x_2 \) (black), \( x_3 \) (green), and \( x_4 \) (red). Since by the eigenvectors’ construction all the amplitudes of the remaining SVF beads are equal to zero, Eq. (26) reads for \( n = 2 \):

\[-\tau_0 \dot{x}_1 = \mu_1 x_1 + (f - 2) \rho_1 x_1 + v_1 x_2 + \rho_1 x_3,
-\tau_0 \dot{x}_2 = (f - 1) v_1 x_1 + \mu_3 x_2 + v_2 x_3 + \rho_2 x_4,
-\tau_0 \dot{x}_3 = (f - 1) \rho_1 x_1 + v_2 x_2 + \mu_4 x_3 + v_3 x_4,
-\tau_0 \dot{x}_4 = \rho_2 x_2 + v_3 x_3 + \mu_4 x_4 - \rho_1 x_4.\]  

(41)

In Eq. (41) we made use of Eqs. (15)–(25). Determining the eigenvalues of this system of equations is equivalent to diagonalizing its matrix of coefficients:

\[A_2 = \begin{pmatrix}
\mu_1 + (f - 2) \rho_1 & v_1 & \rho_1 & 0 \\
(f - 1) v_1 & \mu_3 & v_2 & \rho_2 \\
(f - 1) \rho_1 & v_2 & \mu_4 & v_3 \\
0 & \rho_2 & v_3 & \mu_4 - \rho_1
\end{pmatrix}.\]  

(42)

We call \( A_2 \) the “reduced” matrix of the group \( n = 2 \). One should note that here and in what follows the last diagonal element corresponds to the root bead of the whole leaf under consideration.

In the same way, the structure of the reduced \( F_1(n) \times F_1(n) \) matrix for \( 3 \leq n \leq g \), in which two leaves \( L_1^{(n)} \) move against each other, turns out to be

\[A_n = \begin{pmatrix}
L_{n-1} & C_{12} & C_{13} & 0 \\
C_{21} & \mu_6 & C_{23} & C_{24} \\
C_{31} & C_{32} & \tilde{A}_{n-1} & C_{34} \\
0 & 0 & C_{43} & \tilde{L}_{n-1}
\end{pmatrix}.\]  

(43)

whose details are discussed in Appendix A. In Eq. (43) the matrices \( L_{n-1}, \tilde{A}_{n-1}, \) and \( \tilde{L}_{n-1} \) represent the three leaves \( L_1^{(n-1)}, L_2^{(n-1)}, \) and \( L_2^{(n-1)} \). Let us focus on a leaf \( L_2^{(n)} \) described by \( \tilde{A}_n \). Pictorially, the inclusion of an additional bond into \( L_1^{(n)} \), see Fig. 7(b), changes the character of one of the blue \( L_1^{(n-1)} \) leaves of Fig. 7(a) to a \( L_2^{(n-1)} \) leaf, depicted now in brown in Fig. 7(b). The new, brown leaf introduces \( F_2(n - 1) \) new variables into \( \tilde{A}_n \), hence the rank of \( \tilde{A}_n \) is larger than that of \( A_n \) by \( F_2(n - 1) \).

Consider now exemplarily the matrix \( \tilde{A}_2 \); compared to \( A_2 \) its rank grows by \( F_2(1) = 1 \), being hence a \( 5 \times 5 \) matrix. Furthermore, we have explicitly

\[\tilde{A}_2 = \begin{pmatrix}
\mu_4 & (f - 2) \rho_1 & v_2 & \rho_1 & 0 \\
\rho_1 & \mu_1 + (f - 3) \rho_1 & v_1 & \rho_1 & 0 \\
v_2 & (f - 2) \rho_1 & \mu_5 & v_2 & \rho_2 \\
\rho_1 & (f - 2) \rho_1 & v_2 & \mu_4 & v_3 \\
0 & 0 & \rho_2 & v_3 & \mu_4 - \rho_1
\end{pmatrix},\]  

(44)

as can be determined by the inspection of \( L_2^{(2)} \), indicated by the solid line in Fig. 2, together with Eqs. (15)–(25). The iterative construction of the matrices \( A_n \) and \( \tilde{A}_n \) for \( n \geq 3 \) is relegated to Appendix A.

We turn now to the dynamics of the group \((g + 1)\) of eigenvectors, in which the core also moves. Here, beads which are symmetric with respect to the core move with identical amplitudes. Again, we consider situations of growing \( g \).

The case \( g = 1 \) was already mentioned in Sec. V, where we obtained two independent EM, see Eq. (38). To these EM corresponds the reduced matrix

\[B_1 = \begin{pmatrix}
\mu_1 + (f - 1) \rho_1 & v_1 & f v_1 \\
0 & \mu_2
\end{pmatrix}.\]  

(45)

Similarly, for \( g = 2 \) we find

\[B_2 = \begin{pmatrix}
\mu_1 + (f - 2) \rho_1 & v_1 & \rho_1 & 0 & 0 \\
(f - 1) v_1 & \mu_3 & v_2 & \rho_2 & 0 \\
(f - 1) \rho_1 & v_2 & \mu_4 & v_3 & \rho_2 \\
0 & \rho_2 & v_3 & \mu_4 + (f - 1) \rho_1 & v_2 \\
0 & 0 & f v_2 & f v_2 & \mu_6
\end{pmatrix},\]  

(46)

as can be read off from Fig. 2 together with Eqs. (15)–(25).

Increasing \( g \) does not change the basic structure of \( B_g \), which is a \((F_1(g) + 1) \times (F_1(g) + 1)\) matrix. One has

\[B_g = \begin{pmatrix}
\hat{L}_g & \hat{C}_{1,2} \\
\hat{C}_{2,1} & \mu_6
\end{pmatrix}.\]  

(47)

Here \( \hat{L}_g \) corresponds to the \( f \) leaves \( L_1^{(g)} \) attached to the core; \( \hat{C}_{1,2} \) and \( \hat{C}_{2,1} \) describe the interactions between them. The explicit realization of \( B_g \) for general \( g \) is discussed in Appendix B.

The limit \( q_2 \to 0 \) and \( q_1 \to 0 \) corresponds to a fully flexible macromolecule. In this limit one has \( \rho_1 \equiv \rho_2 = 0, v_1 \equiv v_2 \equiv v_3 = -1, \mu_1 = 1, \mu_4 = 2, \) and \( \mu_2 \equiv \mu_3 \equiv \mu_5 \equiv \mu_6 = f \). This additional simplification allows to determine the VF eigenvalues in terms of recurrent polynomials.\(^{10}\)

For SVF the direct diagonalization of \( A^{\text{SVF}} \), which is a \( N \times N \) matrix, is limited to quite low values of \( f \) and \( g \), see the dependence of \( N \) on \( f \) and \( g \) given in Table I. On the other hand, the procedure outlined above requires only the diagonalization of the reduced matrices \( A_n \), Eq. (43), and \( B_g \), Eq. (47), whose maximal dimension is \((F_1(g) + 1) \times (F_1(g) + 1)\). Now, \( F_1(g) + 1 \) is first, independent of \( f \), and second, much smaller than \( N \). In this way we are able to determine the eigenvalues of \( A^{\text{SVF}} \) for arbitrary \( f \) and reasonable \( g \).
(easily up to \( g = 8 \)) very precisely. Furthermore, as discussed in Sec. V, for each eigenvalue its degeneracy follows automatically from the structure of its eigenmodes.

A further advantage of the procedure shows up when one has to determine the spectra of a whole series of SVF as a function of growing \( g \). Namely, by going from \( g \) to \(( g + 1)\), only one new \( A_{g+1} \) matrix and the new matrix \( B_{g+1} \) appear, while the matrices \( A_1, \ldots , A_g \) stay unchanged. Thus, the step \( g \rightarrow (g + 1) \) requires the additional diagonalization of \( A_{g+1} \) and of \( B_{g+1} \) only.

VII. DISCUSSION

Given the diagonalization procedure of the last section, we determined the full eigenvalue spectra \( \{\lambda_i\} \) of several \( A^{SVF} \) as a function of the stiffness degrees \( q_2 \) and \( q_f \) for several \( f \) and reasonably large \( g \). We found that the \( A^{SVF} \) spectra are very sensitive to \( q_2 \) and \( q_f \), fact which reflects itself in the mechanical moduli. Here, we present exemplarily the results for SVF with \( g = 8 \) and \( f = 4 \); we set \( q_f = q_2/(f - 1) \), by which the stiffness parameters depend on the single variable \( q_2 \). This choice allows to reach both for \( q_2 \) and for \( q_f \) the fully flexible and the stiff limits simultaneously, see Refs. 17 and 18.

The spectra determined in this way enable now the evaluation of the corresponding reduced loss moduli \( [G''(\omega)] \), see Eq. (14). We focus on the \( [G''(\omega)] \), since they have a richer structure than the \([G'(\omega)] \). These loss moduli are presented in Fig. 8: The upper part of the figure displays in double logarithmic scales \([G''(\omega)] \), while in the lower part the slopes \( \alpha(\omega) = d \ln[G''(\omega)]/d \ln \omega \) of the \([G''(\omega)] \)-curves are shown in simple logarithmic scales.

Displayed are the curves for SVF with \( f = 4 \) and \( g = 8 \), plotted for \( q_2 = 0.00, 0.33, 0.66, \) and \( 0.99 \). One can easily remark that with increasing stiffness the domain of the \([G''(\omega)] \) gets broader. This agrees with the general finding for many semiflexible structures. Moreover, the \([G''(\omega)] \)-curves obey their standard scaling patterns, namely \([G''(\omega)] \sim \omega \) for very small frequencies, \( \alpha t_0 \ll 1 \), and \([G''(\omega)] \sim \omega^{-1} \) for very high ones, \( \alpha t_0 \gg 1 \). The particular topology of the VF reveals itself in the intermediate frequency domain. In this domain one observes scaling, in that the overall pattern is rather well reproduced by straight lines, whose slopes are close to the value 0.59 for all \( q_2 \). As shown in Ref. 10 for VF, \( \tilde{d}/2 \), where \( \tilde{d} \) is the spectral dimension, provides a very good theoretical estimate for the \( \alpha \) slope. One has in general, as a function of \( f, \tilde{d} = 2 \ln(f + 1)/ \ln(3f + 3) \), which leads for \( f = 4 \) to \( \tilde{d}/2 \simeq 0.5943 \). Interestingly, this value is also approximately fulfilled for all the SVF displayed in Fig. 8. This fact shows that the global SVF fractal properties are not much influenced by the stiffness conditions, which act rather locally.

A more detailed analysis of the curves in the intermediate regime can be performed using the lower part of Fig. 8, which displays \( \alpha(\omega) \). The \([G''(\omega)] \)-curves show a slightly wavy pattern which reflects the hierarchy, i.e., the value of \( g \), of the particular SVF under investigation. The \( \alpha(\omega) \)-curves enhance this aspect, the number of local maxima being directly related to the value of \( g \). We also remark that in a quite extended \( \omega \)-region the \( \alpha(\omega) \) oscillate nicely between the values 0.5 and 0.67. In fact, one may also go as far as to determine a mean value \( \overline{\alpha} \) for these \( \alpha(\omega) \); we find \( \overline{\alpha} \cong 0.592, 0.591, 0.590, \) and 0.590 for \( q_2 = 0.00, 0.33, 0.66, \) and 0.99, respectively. In general, it turns out that for SVF the domain of intermediate scaling grows with the generation \( g \), while the corresponding slope depends on the functionality \( f \) via \( \tilde{d} \), exactly as in the fully flexible VF case.

Another feature which becomes evident from the upper curve of Fig. 8, is that with increasing \( q_2 \) the maximum of \([G''(\omega)] \) drifts to higher \( \omega \)-values; this leads to the appearance of a second, local maximum for larger \( q_2 \). These findings are, of course, also reflected in the values of \( \alpha(\omega) \).

It is interesting to compare the results obtained for SVF to similar findings for semiflexible dendrimers (SD). In Fig. 9 we show the \([G''(\omega)] \) and the \( \alpha \)-values for SD of different stiffness. For this we use the method employed in Ref. 30. In order to keep the SVF- and the SD-parameters comparable, we choose \( f = 4 \) also for the SD and take for the SD generation \( g = 11 \), since then the number of SD beads is \( N = 354293 \), quite close to the value \( N = 390625 \) which holds for a SVF of generation \( g = 8 \) and \( f = 4 \). For SVF we had as \( q_2 \) stiffness values 0.00, 0.11, 0.22, and 0.33, recalling that \( q_4 = q_2/3 \); for the SD we take a set of very close \( q \)-values, namely 0.00, 0.11, 0.21, and 0.32. We recall that dendrimers have only two kinds of beads, of functionalities \( \phi = 1 \) and \( \phi = f \) (here \( f = 4 \)), so that for SD only one single stiffness parameter \( q \) appears.

---

**FIG. 8.** Reduced loss moduli \([G''(\omega)] \) (upper part) and the corresponding slopes \( \alpha(\omega) \) (lower part) for a VF of functionality \( f = 4 \) and generation \( g = 8 \), plotted for different degrees of stiffness, see text for details.
minded the full set of eigenvalues of the EM by diagonalizing the corresponding matrix of coefficients $A^{SVF}$. The knowledge of this full set of eigenvalues is paramount for being able to compute the dynamical characteristics of many observables, such as the mechanical relaxation moduli.

Now, the size of the matrix $A^{SVF}$ grows very rapidly with $g$ and $f$; this renders prohibitive the brute-force diagonalization of $A^{SVF}$ for large values of the $(g, f)$ pair. The inherent symmetry of VF allows us, however, to considerably simplify the numerical effort through the construction of a particular, complete set of eigenmodes. Using this set, the diagonalization of $A^{SVF}$ reduces to the diagonalization of a series of much smaller matrices. We showed that the matrices which have to be considered are maximally of size $(F_1(g) + 1) \times (F_1(g) + 1)$, where $F_1(g)$ is given by Eq. (37). Moreover, the dimension of these matrices is independent of $f$, by which SVF of the same $g$ but different $f$ can be treated on the same footing, with the same effort. Another advantage of the scheme presented here is that it provides automatically the eigenvalues’ degeneracies; one would be confronted to a very difficult task, had one to determine the degeneracies from a huge list of eigenvalues, whose numerical expressions are not exact, but blurred by slight errors.

Based on the eigenvalue spectra we calculated the mechanical loss moduli. As found for other structures, such as polymer-stars and dendrimers, \(^1^8\) under growing stiffness the moduli broaden and under quite stiff conditions they may show new local maxima. On the other hand, in the intermediate frequency domain the loss moduli for semiflexible VF inherit the average scaling-laws pattern and the wavy behavior of the fully flexible VF case.

Finally, through this work we have demonstrated that the STP-scheme is very well suited for treating complex, semiflexible, branched architectures on a semi-analytical level. Moreover, we could show that the analytic formulation of the EM provides much help in investigating polymeric structures of high topological symmetry.

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APPENDIX A: ITERATIVE CONSTRUCTION OF THE $A_n$ AND $\tilde{A}_n$ MATRICES

While being aware of other implicit possibilities to determine $A_n$ for arbitrary $n$, we present here an explicit algorithm for its iterative construction. In general, the matrices $A_{n+1}$ and $\tilde{A}_{n+1}$ can be obtained from $A_n$ and $\tilde{A}_n$ using only simple modifications, which are also easy to implement in a numerical program. Given that we explicitly presented in Sec. VI the
matrices $A_n$ and $\tilde{A}_n$, the procedure described below paves the way to construct step-by-step $A_n$ and $\tilde{A}_n$ for $n \geq 3$.

1. Determination of $A_{n+1}$

The general form of the $F_1(n+1) \times F_1(n+1)$ matrix $A_{n+1}$ is

$$A_{n+1} = \begin{pmatrix} L_n & C_{12} & C_{13} & 0 \\ C_{21} & \mu_6 & C_{23} & 0 \\ C_{31} & C_{32} & \tilde{A}_n & C_{34} \\ 0 & 0 & C_{43} & L_n \end{pmatrix}. \quad (A1)$$

In Eq. (A1) the notation follows Fig. 7(a): The matrices $L_n$, $\tilde{A}_n$, and $L_n$ represent the three leaves $L_1^{(n)}$, $L_2^{(n)}$, and $L_3^{(n)}$ of Fig. 7(a), and the single diagonal element $\mu_6$ stands for the glue bead. Furthermore, the off-diagonal blocks $C_{xy}$ represent the interactions between these distinct parts. Since in our model only NN and NNN interactions are present and since we consider groups with $n \geq 2$, the matrices which stand for the interactions between the blue $L_1^{(n)}$ leaves and the red $L_2^{(n)}$ leaf as well as for the interactions between the glue bead and $L_2^{(n)}$ vanish.

Now, the $A_{n+1}$ matrix is ordered such that it starts from the blue $L_1^{(n)}$ leaves described by $L_n$, via the sole glue bead, to the green $L_2^{(n)}$ leaf, described by $\tilde{A}_n$, and to the red $L_2^{(n)}$ leaf, described by $L_n$, of Fig. 7(a). Recalling that the root bead of the leaf under consideration is always represented by the last diagonal element of the corresponding matrix, $k = F_1(n)$ is the index of the root bead of a blue $L_1^{(n)}$ leaf in $L_n$ and $m = F_2(n)$ is the index of the root bead of the green $L_2^{(n)}$ and of the red $L_2^{(n)}$ leaf in $\tilde{A}_n$ and in $L_n$, respectively.

As we proceed to show, the diagonal blocks of $A_{n+1}$ can be readily constructed from $A_n$ and $\tilde{A}_n$, while the off-diagonal blocks are known in closed form.

Now, $L_n$ is very similar to $A_n$, from which it differs only in the last diagonal element $k$, since $A_n$ describes the motion of two $L_1^{(n)}$ leaves moving against each other, while $L_n$ reflects the equivalent movement of $(f - 1) L_1^{(n)}$ leaves. Thus

$$(L_n)_{ij} = (A_n)_{ij} + (f - 1) \rho_1 \delta_{ik} \delta_{kj}. \quad (A2)$$

Proceeding with $\tilde{A}_n$, we remark that the corresponding $L_2^{(n)}$ leaf has the same geometry as $L_2^{(n)}$, but that its root bead, indexed by $m$, has an oppositely moving NNN. By inspection, it follows that $\tilde{L}_n$ is related to $\tilde{A}_n$ through

$$(\tilde{L}_n)_{ij} = (\tilde{A}_n)_{ij} - \rho_1 \delta_{im} \delta_{mj}. \quad (A3)$$

The non-diagonal blocks of $A_{n+1}$ stem from beads belonging to different sets (distinct leaves or the glue bead). Thus, the blocks $C_{12}$ and $C_{21}$ are due to the interactions between the glue bead of $L_1^{(n+1)}$ and the blue $L_1^{(n)}$ leaves, see Eq. (A1) and Fig. 7(a). Now, the root bead of $L_1^{(n)}$ and its NN in $L_1^{(n)}$ are the NN and NNN of the glue bead, respectively. Thus the $F_1(n) \times 1$ block $C_{12}$ has only two non-vanishing elements:

$$(C_{12})_{ij} = \rho_2 \delta_{i,k-1} + \nu_2 \delta_{ik}. \quad (A4)$$

where we set $k = F_1(n)$ as above. For the $1 \times F_1(n)$ block $C_{21}$ we have a similar picture but from the point of view of the glue bead, which has $(f - 1)$ root beads as NN and their $(f - 1)$ NN of the $(f - 1)$ attached $L_1^{(n)}$ leaves as NNN. Hence, the entries of $C_{21}$ differ from those of $C_{12}$ by a factor of $(f - 1)$:

$$(C_{21})_{ij} = (f - 1) \rho_2 \delta_{j,k-1} + (f - 1) \nu_2 \delta_{jk}. \quad (A5)$$

Furthermore, the $F_1(n) \times F_2(n)$ block $C_{13}$ and the $F_2(n) \times F_1(n)$ block $C_{31}$ represent the interactions between the leaves $L_1^{(n)}$ and $L_2^{(n)}$, see Eq. (A1). Due to the VF geometry, only a single NNN interaction must be accounted for, namely that between the root bead of a leaf $L_1^{(n)}$ and the root bead of leaf $L_2^{(n)}$. This leads to one single non-vanishing entry in $C_{13}$:

$$(C_{13})_{ij} = \rho_1 \delta_{ik} \delta_{mj}. \quad (A6)$$

where $k = F_1(n)$ and $m = F_2(n)$ as above. On the other hand, $C_{31}$ accounts for the $(f - 1)$ NNN interactions between the root bead of leaf $L_2^{(n)}$ and the $(f - 1)$ root beads of the $(f - 1)$ leaves $L_1^{(n)}$. Thus $C_{31}$ reads

$$(C_{31})_{ij} = (f - 1) \rho_1 \delta_{im} \delta_{kj}. \quad (A7)$$

The $1 \times F_2(n)$ block $C_{23}$ and the $F_2(n) \times 1$ block $C_{32}$ account for the interactions of the glue bead with $L_2^{(n)}$; these involve the root of $L_2^{(n)}$ and its NN bead. We find that $C_{23} = C_{32}^T$ and that

$$(C_{23})_{ij} = (C_{32})_{ij} = \nu_2 \delta_{im} + \rho_2 \delta_{i,m-1}. \quad (A8)$$

In a similar way, the $F_2(n) \times F_2(n)$ blocks $C_{34}$ and $C_{43}$ are symmetric, $C_{34} = C_{43}^T$, since the leaves $L_2^{(n)}$ and $L_2^{(n)}$ are connected through a single bond back to back to each other and we have to take only NN and NNN interactions into account. It turns out that

$$(C_{34})_{ij} = \rho_2 (\delta_{i3} \delta_{1j} + \delta_{13} \delta_{ij}) + \nu_2 \delta_{i1} \delta_{1j}. \quad (A9)$$

2. Determination of $\tilde{A}_{n+1}$

The matrix $\tilde{A}_{n+1}$ corresponds to a $L_2^{(n+1)}$ leaf and is thus of size $F_2(n + 1) \times F_2(n + 1)$. Using the $L_2^{(n+1)}$ structure introduced in Sec. VI, the corresponding matrix $\tilde{A}_{n+1}$ can be iteratively expressed in the following way:

$$\tilde{A}_{n+1} = \begin{pmatrix} \tilde{A}_n & D_{n,n+1} \\ D_{n+1,n} & E_{n+1} \end{pmatrix}. \quad (A10)$$

The matrix $\tilde{A}_{n+1}$ is ordered such that it starts from the brown $L_2^{(n)}$ leaf described by $\tilde{A}_n$ and continues to the other parts of
for the different situations for the root bead of a glue bead of the brown and green leaves, as well as the equivalently moving NN and NNN of the glue bead, see Fig. 7(b). In summary, one has

$$ (E_{n+1})_{ij} = (A_{n+1})_{ij} + \rho_2 \delta_{i,k} \delta_{jk} - [\rho_1 \delta_{i,l-1} + v_2 \delta_{il} + \rho_1 \delta_{i,l+m} \delta_{i,l+1,j} - \rho_2 \delta_{i,l} \delta_{i,l-2,j}] \quad \text{Equation (A11)} $$

The second term of the right-hand side of Eq. (A11) accounts for the different situations for the root bead of a $L_1^{(n+1)}$ and of a $L_1^{(n)}$ leaf, while the third and fourth terms reflect the changes in the number of NN and NNN due to having one blue $L_1^{(n)}$ leaf less.

The two blocks $D_{n,n+1}$ and $D_{n+1,n}$ are very unusual (each of them has only four non-trivial entries, since in our model only NN and NNN interact) and represent the interactions within a $L_2^{(n+1)}$ leaf. Hence they are $F_2(n) \times F_1(n+1)$ and $F_1(n+1) \times F_2(n)$ matrices, respectively. Recalling that the $l$th column and row in $E_{n+1}$ is related to the glue bead of $L_2^{(n+1)}$, its interactions with the brown leaf in Fig. 7(b) are represented by the $l$th column in $D_{n,n+1}$ and by the $l$th row in $D_{n+1,n}$. Hence, by again using $l = F_1(n+1)$ and $m = F_2(n)$ for the root beads of the brown and green $L_2^{(n)}$ leaves we have

$$ (D_{n,n+1})_{ij} = [(f-2)\rho_1 \delta_{i,l-1} + v_2 \delta_{i,l} + \rho_1 \delta_{i,l+m} \delta_{i,m,j} + \rho_2 \delta_{i,l,m-1} \delta_{ij} \quad \text{Equation (A12)} $$

and

$$ (D_{n+1,n})_{ij} = [(\rho_1 \delta_{i,l-1} + v_2 \delta_{i,l} + \rho_1 \delta_{i,l+m} \delta_{i,m+1} + \rho_2 \delta_{i,l} \delta_{i,l-1,j} \quad \text{Equation (A13)} $$

**APPENDIX B: CONSTRUCTION OF THE MATRICES $B_g$ FOR SVF OF ARBITRARY GENERATION $g$**

Finally, we consider the reduced matrix $B_g$ of the last group of eigenmodes. Recalling Eq. (47), $B_g$ reads

$$ B_g = \begin{pmatrix} \hat{L}_g & \tilde{C}_{12} \\ \tilde{C}_{21} & \mu_6 \end{pmatrix} \quad \text{Equation (B1)} $$

Here, the block $\hat{L}_g$ represents the $f L_1^{(g)}$ leaves which are attached to the core of the whole VF of generation $g$. The relation between $\hat{L}_g$ and $A_g$ is as follows:

$$ (\hat{L}_g)_{ij} = (A_g)_{ij} + f \rho_1 \delta_{ik} \delta_{kj} \quad \text{Equation (B2)} $$

where $k$ is again the index of the root bead of $L_1^{(g)}$, here namely $k = F_1(g)$. The difference between $\hat{L}_g$ and $A_g$ resides in the fact that $A_g$ depicts the motion of two $L_1^{(g)}$ leaves against each other, while $\hat{L}_g$ describes a motion in which the symmetrically equivalent beads of all $f L_1^{(g)}$ attached to the core move in the same way.

The blocks $\tilde{C}_{12}$ and $\tilde{C}_{21}$ represent the interactions of the core with its $f$ attached leaves $L_1^{(g)}$. Thus they are $F_1(g) \times 1$ and $1 \times F_1(g)$ matrices, respectively. Both of them have only two non-vanishing entries which follow from to the NN and NNN interactions with the core. Thus

$$ (\tilde{C}_{12})_{ij} = v_2 \delta_{ik} + \rho_2 \delta_i, k-1 \quad \text{Equation (B3)} $$

and

$$ (\tilde{C}_{21})_{ij} = v_2 \delta_{ijk} + f \rho_2 \delta_{i,k}, k-1 \quad \text{Equation (B4)} $$

Finally, the lower diagonal element $\mu_6$ in Eq. (B1) stems from the core of the VF.