Supplemental Material: Universality at Breakdown of Quantum Transport on Complex Networks

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We show first results of the behavior at breakdown for special networks with loops.

DETERMINISTIC NETWORKS WITH LOOPS

Husimi cacti

A Husimi cactus (HC) is the dual structure of a dendrimer, in which each edge of the dendrimer is replaced through a vertex. The vertices of the HC are directly connected if the corresponding edges of the dendrimer are adjacent. In this way a HC of generation $G$ constructed from a dendrimer of functionality $f$ has

$$N_{\text{H}} = (f(1)^G - 1)/(f - 2)$$

vertices. The internal vertices of a HC have functionality $2(f - 1)$ and the peripheral ones $(f - 1)$.

The structure of the Laplacian spectra of HCs is known [1], and it is based on the eigenmodes which are divided on $G + 1$ groups:

- The first group is related to the eigenmodes in which only two nearest-neighboring peripheral beads move against each other. There are $f(f - 2)(f(1)^G - 2)$ such modes and the corresponding eigenvalue is equal to $E_\ast = f$.

- The second group is related to the eigenvalues $f \pm 2\sqrt{f - 1}$, each of them is $f(f - 2)(f(1)^G - 3)$ fold degenerate.

- The third group contains three distinct eigenvalues, one of them is equal to $f$. Each of the eigenvalues appears $f(f - 2)(f(1)^G - 4)$ times.

- In the $m$th group there are $m$ eigenvalues, each of them has a multiplicity $f(f - 2)(f(1)^G - m - 1)$. If $m$ is an odd number, there is again the eigenvalue $f$ in the set.

- The $G$th group contains $G$ distinct eigenvalues with multiplicity $f - 1$. Again, if $G$ odd, there are $f - 1$ eigenvalues $f$.

- The $(G+1)$st group contains $G$ distinct eigenvalues, one of them is zero. Also here, if $G + 1$ is odd, there is the eigenvalue $f$ in the group.

Summarizing, the most degenerate eigenvalue is $E_\ast = f$, its degeneracy is $(f - 1)^G$. Therefore

$$\rho(E_\ast) = (f(1)^G)/N_{\text{H}}$$

and

$$\chi_\infty = \left(1 - \frac{2}{f}\right)^2.$$

Accounting for all eigenvalues yields

$$\chi_\infty = \left(1 - \frac{2}{f}\right)^2 \left(1 + \frac{2}{f^2 - 2f + 2}\right).$$

Thus

$$\kappa_{\text{HC}} = \lim_{f \to 0} \frac{\ln(1 - \chi_\infty)}{\ln(1/f)} = 1$$

as for dendrimers.

Complete graph based recursive networks

Here we consider complete-graph based networks (CGN) of Ref. [2]. The construction of these networks starts from a complete graph; at each iteration step to each node a new complete graph is attached in the way that the new subgraph and the old one share a node together. If the initial complete graph has $n$ nodes, the resulting network will have

$$N_{\text{CGN}} = n^G$$

vertices at the generation $G$.

The Laplacian spectrum of CGN is as follows [2]: The most degenerate eigenvalue $E_\ast = n$ has degeneracy $n^G - 2n^{G-1} + 1$. Then $2^1$ eigenvalues have each a degeneracy $n^{G-1} - 2n^{G-2} + 1$, etc., $2^m$ eigenvalues have each a degeneracy $n^{G-m} - 2n^{G-m-1} + 1$ until $m = G - 2$ reached. The penultimate group contains $2^{G-1}$ $(n - 1)$-fold degenerate) eigenvalues and the last group has only the nondegenerate eigenvalue 0. In this way we obtain

$$\rho(E_\ast) = 1 - \frac{2}{n} + \frac{1}{n^G}$$

and

$$\chi_\infty = \left(1 - \frac{2}{n}\right)^2.$$

Accounting for the whole spectrum leads to

$$\chi_\infty = \frac{(n - 2)^2}{n^2 - 2}.$$

Thus, both $\varepsilon_{\text{CGN}}$ and $\kappa_{\text{CGN}}$ are equal to one, as for the treelike graphs.